

KUGA-SATAKE VARIETIES AND THE HODGE CONJECTURE

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INTRODUCTION

Kuga-Satake varieties are abelian varieties associated to certain weight two Hodge structures, for example the second cohomology group of a K3 surface. We start with an introduction to Hodge structures and we give a detailed account of the construction of Kuga-Satake varieties. The Hodge conjecture is discussed in section 2. An excellent survey of the Hodge conjecture for abelian varieties is [G].

We point out a connection between the Hodge conjecture for abelian varieties and Kuga-Satake varieties in section 9. In section 10 we discuss the implications of the Hodge conjecture on the geometry of surfaces whose second cohomology group has a Kuga-Satake variety. We conclude with some recent results, inspired by an example of C. Voisin, on Kuga-Satake varieties of Hodge structures on which an imaginary quadratic field acts.

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1. POLARIZED HODGE STRUCTURES

1.1. Definition. A (rational) Hodge structure of weight $k \in \mathbf{Z}$ is a \mathbf{Q} -vector space V with a decomposition of its complexification $V_{\mathbf{C}} := V \otimes_{\mathbf{Q}} \mathbf{C}$:

$$V_{\mathbf{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \text{and} \quad \overline{V^{p,q}} = V^{q,p} \quad (p, q \in \mathbf{Z}).$$

Here complex conjugation on $V_{\mathbf{C}}$ is given by $\overline{v \otimes z} := v \otimes \bar{z}$ for $v \in V$ and $z \in \mathbf{C}$. We will take $k, p, q \geq 0$ except in some of the proofs.

1.2. The (Betti) cohomology groups $H^k(X, \mathbf{Q})$ of a complex smooth projective variety are Hodge structures:

$$H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Identifying $H^k(X, \mathbf{C})$ with harmonic differential forms, the subspaces $H^{p,q}(X)$ consist of the harmonic forms of type (p, q) and $H^{p,q}(X) \cong H^q(X, \Omega^p)$.

1.3. It is useful to identify Hodge structures on V with certain representations of the group \mathbf{C}^* on $V_{\mathbf{R}} := V \otimes_{\mathbf{Q}} \mathbf{R}$. We identify \mathbf{C}^* with a subgroup of $GL(2, \mathbf{R})$:

$$\mathbf{C}^* \cong \left\{ s(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2, \mathbf{R}) : a^2 + b^2 \neq 0 \right\}, \quad z = a + bi \longmapsto s(a, b).$$

The eigenvalues of $s(a, b)$ are $z = a + bi$ and $\bar{z} = a - bi$ with corresponding eigenvectors $e_1 + ie_2$ and $e_1 - ie_2$, here e_j is the standard j -th basis vector of \mathbf{C}^2 . An algebraic representation of \mathbf{C}^* is defined to be a homomorphism

$$h : \mathbf{C}^* \longrightarrow GL(V_{\mathbf{R}})$$

such that, with respect to some basis of $V_{\mathbf{R}}$, the entries of the matrix $h(s(a, b))$ are polynomials, with coefficients in \mathbf{R} , in $a, b, (a^2 + b^2)^{-1}$. The following proposition is well-known:

1.4. Proposition. There is a bijection between rational Hodge structures of weight k on a \mathbf{Q} -vector space V and algebraic representations $h : \mathbf{C}^* \rightarrow GL(V_{\mathbf{R}})$ with $h(t) = t^k$ for $t \in \mathbf{R}$. The Hodge structure defined by h is denoted by (V, h) and its Hodge decomposition is:

$$V^{p,q} := \{v \in V_{\mathbf{C}} : h(z)v = z^p \bar{z}^q v\}.$$

Proof. Composing h with the inclusion $GL(V_{\mathbf{R}}) \subset GL(V_{\mathbf{C}})$ we get a representation of \mathbf{C}^* on $V_{\mathbf{C}}$ and the matrix coefficients of this representation are polynomials in z, \bar{z} and $(z\bar{z})^{-1}$. There is a basis $\{v_i\}$ of $V_{\mathbf{C}}$ of simultaneous eigenvectors: $h(z)v_i = \lambda_i(z)v_i$ for some homomorphisms $\lambda_i : \mathbf{C}^* \rightarrow \mathbf{C}^*$. As λ_i is a polynomial in z, \bar{z} and $(z\bar{z})^{-1}$ we get $\lambda_i(z) = z^p \bar{z}^q$ for some $p, q \in \mathbf{Z}$. Since also the conjugate of an eigenvalue is an eigenvalue (on the conjugated eigenvector) we have a Hodge structure.

Conversely, any element in $V_{\mathbf{C}}$ can be written as $v \otimes 1 + w \otimes i$ with $v, w \in V_{\mathbf{R}}$. If $v \otimes 1 + w \otimes i \in V^{p,q}$ then $v \otimes 1 - w \otimes i \in V^{q,p}$. Let $\{v_r + iw_r\}_r$ be a basis of $V^{p,q}$ with $p \geq q$ and define $V_p = \langle v_r, w_r \rangle_r \subset V_{\mathbf{R}}$ be the span of the v_r, w_r 's. Then

$$V_{\mathbf{R}} = \bigoplus_{p \geq q} V_p, \quad \text{and} \quad V_p \otimes_{\mathbf{R}} \mathbf{C} = V^{p,q} \oplus V^{q,p}.$$

The representation h of \mathbf{C}^* on $V_{\mathbf{R}}$ is constructed on each of the subspaces V_p .

In case $p = q$ we have $V^{p,p} = \overline{V^{p,p}}$, so $V^{p,p} = V_p \otimes \mathbf{C}$. Define $h(a + bi)v := (a^2 + b^2)^p v$ for all $v \in V_p$.

Next fix p, q with $p = q + l$ and $l > 0$ and let $\{v_r + iw_r\}_r$ be a basis for $V^{p,q}$. Then the vectors $\{v_r, w_r\}_r \in V_{\mathbf{R}}$, are independent over \mathbf{R} and $V_p = \bigoplus_r \langle v_r, w_r \rangle$. For each r define a representation of \mathbf{C}^* on $\langle v_r, w_r \rangle \subseteq V_p$ by

$$h(a + bi) = (a^2 + b^2)^q \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^l.$$

The eigenspaces in $V_{\mathbf{C}}$ of this representation are the $V^{p,q}$'s. □

1.5. Tensor products of Hodge structures. We transfer the usual algebra constructions on representations to Hodge structures. Given rational Hodge structures $(V, h_V), (W, h_W)$ of weight k_V, k_W one defines a rational Hodge structure $(V \otimes W, h_V \otimes h_W)$ of weight $k_V + k_W$ by:

$$h_V \otimes h_W : \mathbf{C}^* \longrightarrow GL((V \otimes W)_{\mathbf{R}}), \quad z \longmapsto [v \otimes w \mapsto (h_V(z)v) \otimes (h_W(z)w)].$$

The dual vector space $V^* := Hom_{\mathbf{Q}}(V, \mathbf{Q})$ is also a Hodge structure (of weight $-k_V$):

$$h_V^* : \mathbf{C}^* \longrightarrow GL(V_{\mathbf{R}}^*), \quad (h_V^*(z)f)(v) := f(h_V(z)^{-1}v),$$

here $f \in V_{\mathbf{R}}^* = Hom_{\mathbf{R}}(V_{\mathbf{R}}, \mathbf{R})$ and $v \in V_{\mathbf{R}}$. The Tate Hodge structure $\mathbf{Q}(n)$ ($n \in \mathbf{Z}$) is defined by the vector space \mathbf{Q} and the homomorphism:

$$h_n : \mathbf{C}^* \longrightarrow GL_1(\mathbf{R}), \quad z \longmapsto (z\bar{z})^{-n},$$

it has weight $-2n$ and $\mathbf{Q}(n)^{p,q} = 0$ unless $p = q = -n$ in which case $\mathbf{Q}(n)^{-n,-n} = \mathbf{C}$. We write $V(n) := V \otimes \mathbf{Q}(n)$, it is a Hodge structure of weight $k_V - 2n$ with $V(n)^{p,q} = V^{p+n, q+n}$.

1.6. Morphisms of Hodge structures. A morphism of Hodge structures $f : (V, h_V) \rightarrow (W, h_W)$ is a linear map $f : V \rightarrow W$ such that f intertwines the representations h_V and h_W up to a Tate twist:

$$f(h_V(z)v) = (z\bar{z})^n h_W(z)f(v)$$

(f gives a ‘strict’ morphism of Hodge structures $f : V \rightarrow W(-n)$). A morphism of Hodge structures satisfies $f_{\mathbf{C}}(V^{p,q}) \subset W^{p+n, q+n}$, here $f_{\mathbf{C}}$ is the \mathbf{C} -linear extension of f . Since f commutes with the \mathbf{C}^* -representations, both kernel and image of f are (sub) Hodge structures.

We denote by $Hom_{Hod}(V, W)$ ($\subset Hom_{\mathbf{Q}}(V, W)$) the \mathbf{Q} -vector space of morphisms of Hodge structures.

1.7. Definition. Let V be a rational Hodge structure of weight k and let $h : \mathbf{C}^* \rightarrow GL(V_{\mathbf{R}})$ be the corresponding representation. A polarization on V is a bilinear map:

$$\Psi : V \times V \longrightarrow \mathbf{Q}$$

satisfying (for all $v, w \in V_{\mathbf{R}}$):

$$\Psi(h(z)v, h(z)w) = (z\bar{z})^k \Psi(v, w)$$

and

$\Psi(v, h(i)w)$ is a symmetric and positive definite form:

$\Psi(v, h(i)w) = \Psi(w, h(i)v)$ for all $v, w \in V_{\mathbf{R}}$ and $\Psi(v, h(i)v) > 0$ for all $v \in V_{\mathbf{R}} - \{0\}$.

The map $h(i) : V_{\mathbf{R}} \rightarrow V_{\mathbf{R}}$ is called the Weil operator. A polarization is a strict morphism of Hodge structures $V \otimes V \rightarrow \mathbf{Q}(-k)$.

1.8. Lemma. Let (V, Ψ) be a rational polarized Hodge structure of weight k . We denote the \mathbf{C} -linear extension of Ψ by $\Psi_{\mathbf{C}}$. Then:

1. Ψ is symmetric if k is even, and is alternating if k is odd.
2. For $x_{p,q} \in V^{p,q}$ and $y_{r,s} \in V^{r,s}$:

$$\Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}) = 0 \quad \text{if } (p, q) \neq (s, r).$$

In particular, the direct sum decomposition:

$$V_{\mathbf{R}} = \oplus_p V_p, \quad V_p \otimes_{\mathbf{R}} \mathbf{C} = V^{p,q} \oplus V^{q,p}$$

is an orthogonal direct sum w.r.t. Q .

3. The (restriction of the) \mathbf{C} -bilinear form

$$\Psi_{\mathbf{C}} : V^{p,q} \times V^{q,p} \longrightarrow \mathbf{C}$$

is non-degenerate (so $V^{p,q} \cong_{\Psi_{\mathbf{C}}} (V^{q,p})^{dual}$).

4. If the weight is even, the quadratic form defined by the \mathbf{R} -linear extension of Ψ :

$$Q : V_{\mathbf{R}} \longrightarrow \mathbf{R}, \quad Q(v) := \Psi_{\mathbf{R}}(v, v),$$

satisfies: $(-1)^{l-p} Q|_{V_p} > 0$ where $k = 2l$.

Proof. Note that $h(i)^2 v = h(-1)v = (-1)^k v$ for $v \in V_{\mathbf{R}}$. Now use:

$$\Psi(v, w) = \Psi((h(i)v), h(i)w) = \Psi(w, h(i)^2 v) = (-1)^k \Psi(w, v).$$

Next we observe that for all $z \in \mathbf{C}^*$:

$$(z\bar{z})^k \Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}) = \Psi_{\mathbf{C}}(h(z)x_{p,q}, h(z)y_{r,s}) = \Psi_{\mathbf{C}}(z^p \bar{z}^q x_{p,q}, z^r \bar{z}^s y_{r,s}) = z^{p+r} \bar{z}^{q+s} \Psi_{\mathbf{C}}(x_{p,q}, y_{r,s}).$$

Hence $\Psi_{\mathbf{C}}(x_{p,q}, y_{r,s})$ can be non-trivial only when $p+r = q+s = k$. Since also $p+q = r+s = k$ the second statement follows.

Let $x \in V^{p,p}$, then $x = \lambda v$ with $v \in V_p$, $\lambda \in \mathbf{C}$ and $h(i)v = v$. Then $\Psi_{\mathbf{C}}(x, x) = \lambda^2 \Psi(v, v) = \lambda^2 \Psi(v, h(i)v)$ and $\Psi(v, h(i)v)$ is non-zero if $v \neq 0$. If $p \neq q$ then for any non-zero $x \in V^{p,q}$, $\bar{x} \in V^{q,p}$ and thus $x + \bar{x} \in V_p$ is non-zero. Therefore:

$$\begin{aligned} 0 &< \Psi(x + \bar{x}, h(i)(x + \bar{x})) \\ &= \Psi(x, h(i)\bar{x}) + \Psi(\bar{x}, h(i)x) + 0 + 0 \\ &= 2i^{q-p} \Psi(x, \bar{x}), \end{aligned}$$

in the last step we used the symmetry of $\Psi(x, h(i)y)$. Thus Ψ gives a non-degenerate pairing.

Since $p+q = k = 2l$, $h(i) = i^{p-q} = i^{2p-2l} = (-1)^{p-l}$ on $V^{p,q}$, and also on $V^{q,p}$, hence $h(i)$ acts as $(-1)^{l-p}$ on V_p . As $\Psi(v, h(i)v)$ is positive definite, Ψ is positive definite on V_p if $l-p$ is even and negative definite otherwise. \square

1.9. Example. The cohomology groups $H^k(X, \mathbf{Q})$ (as in 1.2) have a polarization see [W], Théorème IV.7 and corollaire or [GH], p. 123.

2. THE HODGE CONJECTURE

2.1. Hodge cycles. The space of Hodge classes in a rational Hodge structure V of weight k is the \mathbf{Q} -subvector space:

$$B(V) := \begin{cases} 0 & \text{if } k \text{ is odd,} \\ V \cap V^{p,p} & \text{if } k = 2p. \end{cases}$$

Note that $V \hookrightarrow V_{\mathbf{C}} = V \otimes_{\mathbf{Q}} \mathbf{C}$, $v \mapsto v \otimes 1$ and the intersection $V \cap V^{p,p}$ takes place in $V_{\mathbf{C}}$.

2.2. Algebraic cycles. Let X be smooth algebraic variety and let Z be (any) irreducible subvariety of codimension p in X . Then Z defines a cohomology class $[Z] \in H^{2p}(X, \mathbf{Q})$. This defines a cycle class map from the Chow group (with coefficients in \mathbf{Q}) of codimension p cycles

$$[\cdot] : CH^p(X)_{\mathbf{Q}} \longrightarrow H^{2p}(X, \mathbf{Q}), \quad \sum a_i Z_i \longmapsto \sum a_i [Z_i].$$

It is well-known that the image of this map lies in the space of Hodge cycles:

$$[CH^p(X)_{\mathbf{Q}}] \subset B(H^{2p}(X, \mathbf{Q})) = H^{2p}(X, \mathbf{Q}) \cap H^{p,p}(X).$$

2.3. The Hodge conjecture. The Hodge conjecture asserts that:

$$[CH^p(X)_{\mathbf{Q}}] = B(H^{2p}(X, \mathbf{Q}))$$

that is, any cohomology class in $H^{2p}(X, \mathbf{Q})$ which is of type (p, p) is the class of a codimension p cycle.

2.4. The Hodge conjecture is known to be true in case $p = 1$ (it follows from an analysis of the exponential sequence) and thus is also true if $p = -1 + \dim X$ and, obviously also in the cases $p = 0$, $\dim X$. In other cases however the conjecture is very much open.

2.5. Morphisms of Hodge structures and Hodge classes. Let $f : V \rightarrow W$ be a (strict) morphism of Hodge structures. Then $f \in \text{Hom}_{\mathbf{Q}}(V, W) = V^* \otimes W$, which is a Hodge structure. Moreover, we have, for all $z \in \mathbf{C}^*$ and all $v \in V_{\mathbf{R}}$:

$$f(h_V(z)v) = h_W(z)f(v) \quad \text{so} \quad f(v) = h_W(z)f(h_V(z)^{-1}v) = ((h_V^*(z) \otimes h_W(z))f)(v),$$

thus f is an invariant in the representation $h_V^* \otimes h_W$ of \mathbf{C}^* on $(V^* \otimes W)_{\mathbf{R}}$, and hence f is of type $(0, 0)$. (In case one has $f(h_V(z)v) = (z\bar{z})^n h_W(z)f(v)$, f is of type (n, n) .) Thus a morphism of Hodge structures is a Hodge class and one finds that:

$$\text{Hom}_{\text{Hod}}(V, W) \cong B(V^* \otimes W).$$

2.6. Morphisms and correspondences. We apply the relation between morphisms of Hodge structures and Hodge classes to the cohomology of algebraic varieties. Let $V (\subset H^k(X, \mathbf{Q}))$ be a sub-Hodge structure, the orthogonal projection $\pi_V : H^k(X, \mathbf{Q}) \rightarrow V$ (use the polarization on H^k) and the inclusion $i_V : V \hookrightarrow H^k(X, \mathbf{Q})$ are morphisms of Hodge structures. Let $W (\subset H^l(Y, \mathbf{Q}))$ be a sub-Hodge structure isomorphic to V , let $g : V \xrightarrow{\cong} W$. Then

$$f := i_W g \pi_V : H^k(X, \mathbf{Q}) \longrightarrow H^l(Y, \mathbf{Q})$$

is a morphism of Hodge structures and hence $f \in B(H^k(X, \mathbf{Q})^* \otimes H^l(Y, \mathbf{Q}))$.

The cohomology of $X \times Y$ can be computed by the Künneth formula:

$$H^n(X \times Y, \mathbf{Q}) = \bigoplus_{p+q=n} H^p(X, \mathbf{Q}) \otimes H^q(Y, \mathbf{Q}).$$

Poincaré duality shows that $H^p(X, \mathbf{Q}) \cong H^{2d-p}(X, \mathbf{Q})^*$ where $d = \dim X$. Thus f defines a Hodge class in

$$f \in B(H^k(X, \mathbf{Q})^* \otimes H^l(Y, \mathbf{Q})) = B(H^{2d-k}(X, \mathbf{Q}) \otimes H^l(Y, \mathbf{Q})) \hookrightarrow B(H^{2d-k+l}(X \times Y, \mathbf{Q})).$$

According to the Hodge conjecture, there should exist a cycle Z on $X \times Y$ with cycle class $[Z] = f$. A cycle on $X \times Y$ is also called a correspondence between X and Y .

Thus the Hodge conjecture implies that any morphism of Hodge structures has a geometric origin in a cycle Z on the product of the varieties. This observation is of importance for the theory of (the category of Hodge) motives in which, roughly speaking, the objects are sub-Hodge structures and the morphisms are (equivalence classes of) correspondences.

3. THE MUMFORD-TATE GROUP.

3.1. The Mumford-Tate group $MT(V)$ of a rational Hodge structure (V, h) is an algebraic subgroup of $GL(V)$. It allows one to find the Hodge cycles $B(V^{\otimes m})$, for any m , using representation theory of Lie groups.

3.2. As the Hodge structures obtained from smooth projective varieties are polarized, we restrict ourselves to that case. A polarization Ψ on a weight k Hodge structure (V, h) may be considered as an element of $B(V^* \otimes V^*)$ and the isomorphism

$$V \xrightarrow{\cong} V^*, \quad v \longmapsto [w \mapsto \Psi(v, w)]$$

is a morphism of Hodge structures (it is a strict morphism $V \rightarrow V^*(-k)$).

An algebraic group $G (\subset GL(V))$ defined over \mathbf{Q} is defined by polynomial equations in the n^2 matrix entries g_{ij} of g and $\det(g)^{-1}$ with coefficients in \mathbf{Q} . For a commutative \mathbf{Q} -algebra A

(for example $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$), the set of solutions to these equations in A^{n^2} is a group (under matrix multiplication) and this group is denoted by $G(A)$.

The algebraic subgroup of $GL(V)$ ‘fixing’ Ψ is defined to be:

$$G(\Psi) := \{g \in GL(V) : \Psi(gv, gw) = \nu(g)\Psi(v, w) \text{ for some } \nu(g) \in \mathbf{G}_m\}.$$

Here \mathbf{G}_m is the multiplicative group, so $\mathbf{G}_m(\mathbf{Q}) = \mathbf{Q}^*$ etc. In case the weight is odd we usually write $GSp(\Psi)$ and if the weight is even $GO(\Psi)$ for $G(\Psi)$.

3.3. Definition. Let (V, Ψ) be a polarized rational Hodge structure of weight k .

1. G_1 is the algebraic subgroup of the $g \in G(\Psi)$ for which there is a $\omega(g) \in \mathbf{G}_m$ such that $g \cdot t = \omega(g)^m t$ for all $t \in B(V^{\otimes m})$.
2. G_2 is the smallest algebraic subgroup of $GL(V)$ which is defined over \mathbf{Q} and which still satisfies

$$h(\mathbf{C}^*) \subseteq G_2(\mathbf{R}).$$

3.4. Remark. The condition is that $g \in G_1$ acts as a scalar multiple of the identity on the \mathbf{Q} -subspace $B(V^{\otimes m})$ of $V^{\otimes m}$, the shape of the scalar follows from tensoring Hodge cycles. Since $h(\mathbf{C}^*) \in G(\Psi)$ we have $G_2 \subset G(\Psi)$.

3.5. Theorem. For any polarized rational Hodge structure (V, h, Ψ) one has:

$$G_1 = G_2 =: MT(V) \quad \text{and} \quad MT(V) \subset G(\Psi),$$

the algebraic group $MT(V)$ is called the Mumford-Tate group of (V, h, Ψ) . The Hodge cycles in the Hodge structure $(V^{\otimes m}, h^{\otimes m})$ are the $MT(V)$ -‘invariants’:

$$B(V^{\otimes m}) = \{w \in V^{\otimes m} : gw = \omega(g)^m w \quad \forall g \in MT(V)\}.$$

The Mumford-Tate group is reductive (i.e. every finite dimensional representation of $MT(V)$ is a direct sum of irreducible representations).

Proof. This is all in [DMOS], but we defined $MT(V)$ as the projection in $GL(V)$ of the group G in I.3 and we use the polarization to identify V and V^* . Proposition I.3.4 shows that $G_1 = G_2$ and Proposition I.3.1 identifies the Hodge cycles as $MT(V)$ -invariants. The reductivity is proven with Weyl’s unitary trick in Proposition I.3.6, again the polarization is essential. \square

3.6. Corollary. We have: $End_{Hod}(V) \cong End_{MT}(V)$, with

$$End_{MT}(V) = \{M \in End(V) : Mg = gM \quad \forall g \in MT(V)\}.$$

Proof. This follows from Theorem 3.5 since $End(V) = V^* \otimes V \cong V \otimes V$, hence $End_{Hod}(V) \cong B(V^{\otimes 2})$ and $End_{MT}(V)$ corresponds to the space of $MT(V)$ -invariants in $V^{\otimes 2}$. \square

4. HODGE STRUCTURES OF WEIGHT ONE AND TWO

4.1. Given a polarized Hodge structure (V, h, Q) of weight 2, it is interesting to know if there exists a Hodge structure (W, h_W) of weight 1 with

$$V \hookrightarrow W \otimes W$$

(we do not even require that W be polarized). Note that if there is a pair of weight 1 Hodge structures W_1, W_2 with $V \subset W_1 \otimes W_2$ then $V \subset W \otimes W$ with $W = W_1 \oplus W_2$.

In general the existence of W depends very much on the Mumford-Tate group of V but in case $MT(V) = GO(Q)$, the answer is very easy. The proof of the following proposition follows §7, Remarques of [D]. Deligne's observation has been further explored by C. Schoen in [S4].

4.2. **Proposition.** Let (V, h, Q) be a polarized Hodge structure of weight 2 with $\dim V^{2,0} > 0$ and $MT(V) = GO(Q)$.

If $V \subset W \otimes W$ for some Hodge structure W of weight 1 we must have $\dim V^{2,0} = 1$.

Proof. The tangent space at the identity element of an algebraic group is the Lie algebra of that group. The inclusions $SO(Q) \subset GO(Q) = \mathbf{G}_m O(Q) \subset GL(V)$ give inclusion of Lie algebras $so(Q) \subset go(Q) \subset End(V)$. The sub-space $so(Q)$ is defined by the linear equations $\sum_{ij} (\partial F / \partial g_{ij})(I) M_{ij} = 0$ where $F \in \mathbf{Q}[\dots, g_{ij}, \dots]$ runs over the equations defining $SO(Q)$. Since for any $F \in \mathbf{Q}[\dots, g_{ij}, \dots]$ we have $F(\dots, \delta_{ij} + \epsilon m_{ij}, \dots) = F(I) + \epsilon \sum_{ij} (\partial F / \partial g_{ij})(I) M_{ij}$ where $I = (\delta_{ij})$ and $\epsilon^2 = 0$, $so(Q)$ is the subspace of $M \in End(V)$ such that $Q((I + \epsilon M)x, (I + \epsilon M)y) = Q(x, y)$, that is:

$$so(Q) = \{M \in End(V) : Q(Mx, y) + Q(x, My) = 0 \quad \forall x, y \in V\}.$$

Since $h(z) \in GO(Q)$ one has $h(z)SO(Q)h^{-1}(z) \subset SO(Q)$, the differential of this automorphism is called $Ad(h(z))$. It defines a natural Hodge structure on $so(Q)$:

$$Ad(h) : \mathbf{C}^* \longrightarrow GL(so(Q)_{\mathbf{R}}), \quad z \longmapsto Ad(h(z)) = [M \mapsto h(z)Mh(z)^{-1}].$$

Since $h(t) = t^2 I_V$ for $t \in \mathbf{R}^*$, the map $Ad(h(t))$ is the identity, hence $so(Q)$ is a Hodge structure of weight 0.

The spaces $V^{2,0}$ and $V^{0,2}$ are isotropic for Q and Q gives a duality between them, see 1.8. So we can choose a basis f_1, \dots, f_m of $V^{2,0}$ and f_{m+1}, \dots, f_{2m} of $V^{0,2}$ such that $Q(y, y) = y_1 y_{m+1} + y_2 y_{m+2} + \dots + y_m y_{2m}$ with $y = \sum y_i f_i$. In case $m > 1$ the following linear map B lies in $so(Q)_{\mathbf{C}}$:

$$B : V_{\mathbf{C}} \longrightarrow V_{\mathbf{C}}, \quad B(V^{1,1}) = 0, \quad B(f_1) = f_{2m}, \quad B(f_m) = -f_{m+1}, \quad B(f_i) = 0$$

if $i \neq 1, m$. Since $h(z)$ acts as $z^p \bar{z}^q$ on $V^{p,q}$ it is easy to verify that

$$h(z)Bh(z)^{-1} = z^{-2} \bar{z}^2 B, \quad \text{hence} \quad so(Q)^{-2,2} \neq 0.$$

Let W be a Hodge structure of weight 1 and let $h_W : \mathbf{C}^* \rightarrow GL(W_{\mathbf{R}})$ be the homomorphism which defines the Hodge structure on W . The Hodge structure on $W \otimes W$ is defined by $h_2(z)(u \otimes w) = (h_W(z)u) \otimes (h_W(z)w)$. Thus $MT(W \otimes W) \subset GL(W)$ where $GL(W)$ acts on $W \otimes W$ by $A \cdot (u \otimes w) = (Au) \otimes (Aw)$. Therefore the Lie algebra $Lie(GL(W)) = End(W)$ acts on $W \otimes W$ by

$$X \cdot (u \otimes w) = (Xu) \otimes w + u \otimes (Xw) \quad (X \in End(W), u, w \in W).$$

For $X \in \text{End}(W)$ the map $\text{Ad}(h_2(z))(X) (\in \text{End}(W \otimes W))$ is given by:

$$(h_2(z) \cdot X \cdot h_2(z)^{-1}) \cdot (u \otimes w) = (h_W(z)Xh_W(z)^{-1}u) \otimes w + u \otimes (h_W(z)Xh_W(z)^{-1}w),$$

hence $\text{Ad}(h_2(z))(X) = \text{Ad}(h_W(z))(X) (= h_W(z)Xh_W(z)^{-1} \in \text{End}(W))$ acting on $W \otimes W$. The eigenvalues of $h_W(z)$ on $W_{\mathbf{C}} = W^{1,0} \oplus W^{0,1}$ are z and \bar{z} , so the eigenvalues of the map $\text{End}(W)_{\mathbf{C}} \rightarrow \text{End}(W)_{\mathbf{C}}, X \mapsto h_W(z)Xh_W(z)^{-1}$ are $z\bar{z}^{-1}$, 1 , $z^{-1}\bar{z}$. Thus the Hodge structure on $\text{End}(W)$ ($\subset \text{End}(W \otimes W)$) has trivial $(-2, 2)$ -part. Therefore $\text{so}(Q)$ cannot be contained in $\text{End}(W)$. \square

4.3. We consider the polarized weight two Hodge structures (V, h, Q) with $\dim V^{2,0} = 1$ in some more detail. The quadratic form defined by the polarization on $V_{\mathbf{R}}$ is negative definite on V_2 , a two dimensional subspace, and positive definite on V_1 (cf. 1.8).

4.4. **Lemma.** Let (V, Q) be a \mathbf{Q} -vector space of dimension n with a bilinear form Q of signature $(2-, (n-2)+)$. Then there is a natural bijection between the following two sets

1. The set of algebraic homomorphisms $h : \mathbf{C}^* \rightarrow \text{GO}(Q)(\mathbf{R})$ such that (V, h, Q) is a polarized Hodge structure of weight two.
2. The set of oriented two-dimensional subspaces $W \subset V_{\mathbf{R}}$ such that the restriction of Q to W is negative definite.

Proof. Given h , one defines $W := V_2$ and the orientation on V_2 is defined by the basis $v, h(e^{\pi i/4})v$, for (any) $v \in V_2, v \neq 0$.

Conversely, given the oriented W , define $h(e^{i\phi})$ to be rotation with angle 2ϕ (in the positive sense) on W and to be the identity on W^{\perp} . For $t \in \mathbf{R}$, let $h(t)$ be scalar multiplication by t^2 on V , this defines the representation $h : \mathbf{C}^* \rightarrow \text{GL}(V_{\mathbf{R}})$. \square

4.5. **Existence of polarized weight two Hodge structures.** The Lemma implies that given a \mathbf{Q} -vector space V of dimension n and a non-degenerate quadratic form Q on V with signature $(2-, (n-2)+)$ there exist polarized weight two Hodge structures (V, h, Q) since we can certainly find a $W \subset V_{\mathbf{R}}$ on which Q is negative definite.

There is a basis of $V_{\mathbf{R}}$ such that $Q = -X_1^2 - X_2^2 + X_3^2 + \dots + X_n^2$. If the restriction of Q to a subspace W is negative definite then we can find a basis a, b of W such that $a = (1, 0, a_3, \dots, a_n) = (1, 0, a')$, $b = (0, 1, b_3, \dots, b_n) = (0, 1, b')$ (else W contains a non-zero element $(0, 0, c_3, \dots, c_n)$ contradicting that Q is negative definite on W). Thus W depends on $2(n-2)$ -real parameters $(a', b') \in U \subset \mathbf{R}^{2(n-2)}$ for a certain open subset U .

4.6. **Remark.** The group $\text{GO}(Q)(\mathbf{R})$ acts in a natural way on both of the sets mentioned in Lemma 4.4:

$$h \longmapsto h^g := [z \mapsto gh(z)g^{-1}], \quad W \longmapsto gW,$$

this action is compatible with the bijection we indicated. By Witt's theorem, for any two subspaces $W, W' \in V_{\mathbf{R}}$ on which Q is negative definite, there exists a $g \in \text{SO}(Q)(\mathbf{R})$ with $gW = W'$ ([La], I.4.2). Moreover, if $\dim V > 2$ we can also find a $g \in \text{SO}(Q)(\mathbf{R})$ with $gW = W$ and which reverses the orientation on W . Hence $\text{SO}(Q)$ acts transitively on the set of Hodge structures and this set is thus identified with $\text{SO}(Q)(\mathbf{R})/S(O(2) \times O(n-2))(\mathbf{R})$ where $n = \dim V > 2$. This set is actually the disjoint union (corresponding to the choice of

orientation) of two copies of a Hermitian symmetric domain, in particular, the W depend on $n - 2$ complex parameters.

4.7. Lemma. Let (V, h, Q) be a weight two Hodge structure with $\dim V^{2,0} = 1$. Then, for general h , we have $MT(V) = GO(Q)$.

Proof. The proof is similar to the proof of [vG], 6.11. □

5. FROM WEIGHT TWO TO WEIGHT ONE.

5.1. We recall the construction of Kuga and Satake ([KS], [S1] and [D]) which associates to a polarized Hodge structure (V, h, Q) of weight two, with $\dim V^{2,0} = 1$, a polarized Hodge structure $(C^+(Q), h_s, E)$ of weight one. In Proposition 6.3 we show that one can recover the Hodge structure on V from the one on $C^+(Q)$ and that $V \subset C^+(Q) \otimes C^+(Q)$ (inclusion of Hodge structures).

5.2. Quadratic Forms. Let (V, h, Q) be a polarized weight two rational Hodge structure with $\dim V^{2,0} = 1$, let $n = \dim V$. We will simply write $Q(v)$ for $Q(v, v)$, thus the polarization Q is also viewed as a (non-degenerate) quadratic form Q on the \mathbf{Q} -vector space V . Since Q has signature $(2-, (n-2)+)$ there is a basis of V such that ([La], Cor. 2.4):

$$Q : \quad d_1 X_1^2 + d_2 X_2^2 + \dots d_n X_n^2, \quad d_1, d_2 < 0, \quad d_3, \dots, d_n > 0.$$

5.3. Clifford algebras. Associated to (V, Q) there is a 2^n -dimensional associative \mathbf{Q} -algebra, the Clifford algebra $C(Q)$ (cf. [La], Ch. V, [S2], Ch. 9) and a linear injective map $i : V \hookrightarrow C(Q)$. This is characterized by the following universal property:

Let A be a \mathbf{Q} -algebra and let $f : V \rightarrow A$ be a linear map such that $f(v)^2 = Q(v)$ for all $v \in V$. Then there is a unique \mathbf{Q} -algebra homomorphism $g : C(Q) \rightarrow A$ such that $f = g \circ i$.

We observe that if V has basis e_1, \dots, e_n and Q is given by $\sum d_i X_i^2$ with respect to this basis, one has $e_i^2 = Q(e_i) = d_i$ and, for $i \neq j$, $(e_i + e_j)^2 = Q(e_i + e_j) = d_i + d_j$ whence:

$$e_i^2 = d_i, \quad e_i e_j = -e_j e_i \quad \text{if } i \neq j.$$

A \mathbf{Q} -basis for $C(Q)$ is given by the products

$$e^a := e_1^{a_1} e_2^{a_2} \dots e_n^{a_n}, \quad a = (a_1, \dots, a_n) \in \{0, 1\}^n$$

and

$$C(Q) := \bigoplus_a \mathbf{Q} e^a.$$

In particular, the vector space V is a subspace of $C(V)$:

$$i : V \hookrightarrow C(V), \quad e_i \longmapsto e_1^0 \dots e_i^1 \dots e_n^0,$$

note that V is not a subalgebra however. The even Clifford algebra is the sub-algebra:

$$C^+(Q) := \bigoplus_a \mathbf{Q} e^a, \quad a \in \{0, 1\}^n, \quad \sum a_i \equiv 0 \pmod{2}.$$

Similarly, the vector space $C^-(Q)$ is the span of the e^a with $\sum a_i \equiv 1 \pmod{2}$. We will study the Clifford algebra in more detail in section 7.

5.4. The complex structure on $C^+(Q)_{\mathbf{R}}$. Let

$$h : \mathbf{C}^* \longrightarrow GO(Q)(\mathbf{R}) \subset GL(V_{\mathbf{R}})$$

be the homomorphism defining the Hodge structure on V . Recall that

$$V_{\mathbf{R}} = V_2 \oplus V_1, \quad \text{with } V_1 \otimes \mathbf{C} = V^{1,1}, \quad V_2 \otimes \mathbf{C} = V^{2,0} \oplus V^{0,2},$$

and the direct sum is orthogonal for Q . The space V_2 is a real two dimensional vector space.

5.5. Lemma. Let $\{f_1, f_2\}$ be a basis of V_2 such that $V^{2,0} = \langle f_1 + if_2 \rangle$ and $Q(f_1) = -1$.

1. For all $x, y \in \mathbf{R}$: $Q(xf_1 + yf_2) = -(x^2 + y^2)$.
2. The element $J := f_1f_2 = -f_2f_1 \in C^+(Q)_{\mathbf{R}}$ satisfies $J^2 = -1$.
3. The element J does not depend on the choice of f_1, f_2 as above.

Proof. Let $\langle v + iw \rangle$ be any basis of $V^{2,0}$ with $v, w \in V_2$. The polarization restricted to V_2 is negative definite, hence we can find a $\lambda \in \mathbf{R}$ with $Q(\lambda v) = \lambda^2 Q(v) = -1$, take $f_1 = \lambda v$, $f_2 = \lambda w$. From Lemma 1.8 we have that $Q = 0$ on $V^{2,0}$, hence $0 = Q(f_1 + if_2, f_1 + if_2) = Q(f_1) - Q(f_2) + 2iQ(f_1, f_2)$, hence also $Q(f_2) = -1$ and f_1, f_2 are perpendicular. It follows that $Q(xf_1 + yf_2) = -(x^2 + y^2)$. Thus in $C(Q)_{\mathbf{R}}$ we have: $(xf_1 + yf_2)^2 = -(x^2 + y^2)$ so $f_1f_2 + f_2f_1 = 0$.

In $C(Q)_{\mathbf{R}}$ we now have $f_i^2 = -1$ and $f_1f_2 = -f_2f_1$, therefore

$$J^2 = (f_1f_2)(f_1f_2) = -f_1^2f_2^2 = -1.$$

Let f'_1, f'_2 be another such basis of V_2 , then there is an orthogonal 2×2 matrix A with $f'_i = Af_i$. Since both $f_1 + if_2$ and $f'_1 + if'_2$ span $V^{2,0}$, an eigenspace for all $h(z)$, A must commute with all $h(z)$'s, hence A is a rotation. Therefore $f'_1 = af_1 + bf_2$ and $f'_2 = -bf_1 + af_2$ with $a^2 + b^2 = 1$. Thus $f'_1f'_2 = -ab(-1) + (a^2 + b^2)f_1f_2 + ab(-1) = f_1f_2$. \square

5.6. The weight one Hodge structure on $C^+(Q)$. With J as in Lemma 5.5 we define a homomorphism

$$h_s : \mathbf{C}^* \longrightarrow GL(C^+(Q)_{\mathbf{R}}), \quad a + bi \longmapsto a - bJ := [x \longmapsto (a - bJ)x],$$

(with $a, b \in \mathbf{R}$, $x \in C^+(Q)_{\mathbf{R}}$; the ‘ $-$ ’ sign is needed for 6.3). So we let $a - bJ \in C^+(Q)_{\mathbf{R}}$ act by right multiplication on $C^+(Q)_{\mathbf{R}}$. This is obviously an algebraic homomorphism and defines a rational Hodge structure of weight one on the \mathbf{Q} -vector space $C^+(Q)$.

5.7. Polarizations. We show that the Hodge structure $(C^+(Q), h_s)$ has a polarization. Given $c \in C^+(Q)$ the right multiplication by c is a \mathbf{Q} -linear map $C^+(Q) \rightarrow C^+(Q)$, $x \mapsto cx$. We denote by $Tr(c) (\in \mathbf{Q})$ the trace of this linear map then

$$Tr : C^+(Q) \longrightarrow \mathbf{Q}$$

is a \mathbf{Q} -linear map. There is a \mathbf{Q} -linear algebra anti-involution ι on $C^+(V)$ (so $\iota(xy) = \iota(y)\iota(x)$) which is given by ([La], V, 1.11; [S2], 9.3):

$$\iota : C^+(Q) \longrightarrow C^+(Q), \quad e_1^{a_1} \dots e_n^{a_n} \longmapsto e_n^{a_n} \dots e_1^{a_1} \quad (a_i \in \{0, 1\}).$$

5.8. **Lemma.** Let $\{e^a\}$ with $a \in \{0, 1\}^n$, $\sum a_i \equiv 0 \pmod{2}$, be the standard basis of $C^+(Q)$.

1. We have:

$$Tr(e^a) = \begin{cases} 0 & \text{if } a \neq 0, \\ 2^{n-1} & \text{if } a = 0. \end{cases}$$

In particular:

$$Tr(xy) = Tr(yx) \quad \text{and} \quad Tr(\iota(x)) = Tr(x).$$

2. For $a, b \in \{0, 1\}^n$ we have:

$$Tr(\iota(e^a)e^b) = \begin{cases} 0 & \text{if } a \neq b, \\ 2^{n-1}d_1^{a_1}d_2^{a_2}\dots d_n^{a_n} & \text{if } a = b. \end{cases}$$

Proof. Consider the matrix of multiplication by e^a w.r.t. the standard basis. Since

$$e^a e^b = \lambda e^c, \quad \text{with } c_i \equiv a_i + b_i \pmod{2}$$

and λ is, up to sign, the product of the d_i for which $a_i = b_i = 1$, we see that $e^a e^b$ is a scalar multiple of the basis vector e^c but $e^c = e^b$ only if $a = 0$. Thus $Tr(e^a) = 0$ unless $a = 0$ and then $e^0 = 1$ hence $Tr(e^0) = \dim C^+(Q)$. Since $Tr(e^a e^b) = 0 = Tr(e^b e^a)$ if $a \neq b$ and, obviously, $Tr(e^a e^a) = Tr(e^a e^a)$, we get $Tr(xy) = Tr(yx)$ as $(x, y) \mapsto Tr(xy)$ is bilinear. Finally $\iota(e^a) = \pm e^a$, hence $Tr(\iota(e^a)) = 0 = Tr(e^a)$ if $a \neq 0$ and if $a = 0$ we have $e^a = 1 = \iota(e^a)$.

For the second part, since $\iota(e^a) = \pm e^a$ we have $\iota(e^a)e^b = \mu e^c$ for some $\mu \in \mathbf{Q}$ and $c_i = a_i + b_i \pmod{2}$. Thus $Tr(\iota(e^a)e^b) = 0$ unless $a = b$. In that case $\iota(e^a)e^a = d_1^{a_1} \dots d_n^{a_n}$ since:

$$\iota(e_1^{a_1} \dots e_n^{a_n})e_1^{a_1} \dots e_n^{a_n} = e_n^{a_n} \dots e_1^{a_1} e_1^{a_1} \dots e_n^{a_n} = e_n^{a_n} \dots e_2^{a_2} d_1^{a_1} e_2^{a_2} \dots e_n^{a_n} = \dots = d_1^{a_1} \dots d_n^{a_n}.$$

This concludes the proof of the lemma. \square

5.9. **Proposition.** Let $\alpha := \pm e_1 e_2 \in C^+(\mathbf{Q})$. Then the bilinear form:

$$E : C^+(\mathbf{Q}) \times C^+(\mathbf{Q}) \longrightarrow \mathbf{Q}, \quad E(v, w) := Tr(\alpha \iota(v)w)$$

is a polarization of the weight one Hodge structure $(C^+(\mathbf{Q}), h_s)$ (for suitable choice of sign).

Proof. The form E is obviously \mathbf{Q} -bilinear. We have $J = f_1 f_2 = -f_2 f_1 \in C^+(Q)_{\mathbf{R}}$ hence $\iota(J) = -J$. Thus, with $z = a + bi \in \mathbf{C}^*$ we get:

$$\begin{aligned} E(h_s(z)x, h_s(z)y) &= Tr(\alpha \iota((a - bJ)x)(a - bJ)y) \\ &= Tr(\alpha \iota(x)(a + bJ)(a - bJ)y) \\ &= Tr(\alpha \iota(x)(a^2 + b^2)y) \\ &= z\bar{z}E(x, y). \end{aligned}$$

The symmetry of the bilinear form $E(x, h_s(i)y)$ follows from:

$$\begin{aligned} E(x, h_s(i)y) &= -Tr(\alpha \iota(x)Jy) \\ &= -Tr(\iota(\alpha \iota(x)Jy)) \\ &= -Tr(\iota(y)\iota(J)\iota^2(x)\iota(\alpha)) \\ &= -Tr(\iota(y)(-J)x(-\alpha)) \\ &= Tr(\alpha \iota(y)h_s(i)x) \\ &= E(y, h_s(i)x), \end{aligned}$$

here we used $\iota(\alpha) = \iota(e_1 e_2) = e_2 e_1 = -e_1 e_2 = -\alpha$ and Lemma 5.8.

It remains to check that $E(x, h(i)x)$ is either positive or negative definite. First we show that, given (V, Q) , it suffices to consider just one Hodge structure (V, h, Q) . Let (V, h, Q) and (V, h', Q) be polarized Hodge structures and let $J, J' \in C^+(Q)_{\mathbf{R}}$ be the associated complex structures on $C^+(Q)_{\mathbf{R}}$. From 4.6 we have a $g \in SO(Q)(\mathbf{R})$ with $h' = ghg^{-1}$ and $gV_2 = V'_2$, in particular the $f_1, f_2 \in V_2$ with $J = -f_1f_2 \in C^+(Q)_{\mathbf{R}}$ are mapped to $f'_i = gf_i$ and $J' = -f'_1f'_2$. There is a $\tilde{g} \in C^+(Q)_{\mathbf{R}}$ with $\tilde{g}v\tilde{g}^{-1} = gv$ for all $v \in V$ ($\subset C(Q)$) ([C], Theorem 10.3.1) and $\tilde{g}\iota(\tilde{g}) \in \mathbf{R}^*$ ([C], Cor. 3.1.1, [S2], Lemma 9.3.2). Therefore $\tilde{g}^{-1} = \lambda\tilde{g}$ for a $\lambda \in \mathbf{R}^*$ and $J' = \tilde{g}J\tilde{g}^{-1} = \lambda\tilde{g}J\iota(\tilde{g})$ so:

$$\begin{aligned} E(x, J'x) &= Tr(\alpha\iota(x)J'x) \\ &= \lambda Tr(\alpha\iota(x)\tilde{g}J\iota(\tilde{g})x) \\ &= \lambda Tr(\alpha\iota(y)Jy) \quad \text{with} \quad y := \iota(\tilde{g})x \\ &= \lambda E(y, Jy) \end{aligned}$$

hence $E(x, J'x)$ is definite iff $E(x, Jx)$ is definite.

We consider the Hodge structure with $V_{\mathbf{R}} = V_2 \oplus V_1$, $V_2 = \langle e_1, e_2 \rangle_{\mathbf{R}}$ (so Q restricted to V_2 is $d_1X_1^2 + d_2X_2^2$ with $d_1, d_2 < 0$) and we take $J = ce_1e_2$ with $c = (d_1d_2)^{-1/2} (\in \mathbf{R}_{>0})$. Then we have:

$$E(x, h(i)y) = -Tr(\alpha\iota(x)Jy) = -cTr(e_1e_2\iota(x)e_1e_2y).$$

We write the basis elements as:

$$e^a = e_1^{a_1}e_2^{a_2}f_a \quad \text{with} \quad f_a = e_3^{a_3} \dots e_n^{a_n}.$$

Then $\iota(e^a) = \iota(f_a)e_2^{a_2}e_1^{a_1}$. In case $a_1 = a_2 = 0$ we have:

$$\iota(e^a)e_1e_2 = \iota(f_a)e_1e_2 = (-1)^r e_1\iota(f_a)e_2 = (-1)^{2r}\iota(f_a) = e_1e_2\iota(e^a),$$

here $r = a_3 + \dots + a_n$. Similarly, if $a_1 = 1, a_2 = 0$:

$$\iota(e^a)e_1e_2 = \iota(f_a)e_1e_1e_2 = (-1)^r e_1\iota(f_a)e_1e_2 = -(-1)^r e_1\iota(f_a)e_2e_1 = -e_1e_2\iota(e^a),$$

and proceeding in this way one verifies that: $\iota(e^a)e_1e_2 = e_1e_2(-1)^{a_1+a_2}\iota(e^a)$. Therefore:

$$\begin{aligned} E(e^a, h_s(i)e^b) &= -cTr(e_1e_2\iota(e^a)e_1e_2e^b) \\ &= -cTr((e_1e_2)^2(-1)^{a_1+a_2}\iota(e^a)e^b) \\ &= +cd_1d_2(-1)^{a_1+a_2}Tr(\iota(e^a)e^b). \end{aligned}$$

Using the previous lemma we get:

$$E(e^a, h(i)e^b) = \begin{cases} 0 & \text{if } a \neq b, \\ 2^{n-1}(cd_1d_2)((-1)^{a_1+a_2}d_1^{a_1}d_2^{a_2})d_3^{a_3} \dots d_n^{a_n} & \text{if } a = b. \end{cases}$$

Thus $E(e^a, h(i)e^a) > 0$ for all a since $d_1, d_2 < 0, d_3, \dots, d_n > 0$. This, combined with the \mathbf{Q} -bilinearity of Tr , proves that $E(x, h(i)x) > 0$ for all $x \in C^+(Q) - \{0\}$. \square

6. THE MUMFORD-TATE GROUP OF THE KUGA-SATAKE HODGE STRUCTURE.

6.1. In the previous section we constructed a polarized rational weight one Hodge structure $(C^+(Q), h_s, E)$. We recall some basic facts on this Hodge structure. A detailed study of the Clifford algebra in the next section will give more precise information on the simple sub-Hodge structures of $C^+(Q)$.

6.2. The spin representation. Let Q a non-degenerate quadratic form on a F -vector space V (we usually consider the case $F = \mathbf{Q}$ but the cases $F = \mathbf{R}, \mathbf{C}$ are also of interest to us). The Spin group $CSpin(Q)$, an algebraic group defined over F , can be defined as the subgroup of $C^+(Q)^*$, the units of the ring $C^+(Q)$:

$$CSpin(Q) = \{g \in C^+(Q)^* : gVg^{-1} \subset V\}.$$

By its very definition, we have a homomorphism

$$\rho : CSpin(Q) \longrightarrow GL(V), \quad g \longmapsto [v \mapsto gvg^{-1}].$$

Since $Q(\rho(g)v) = (\rho(g)v)^2 = gv^2g^{-1} = gQ(v)g^{-1} = Q(v)$, the image of ρ lies in $O(Q)$, the orthogonal group of Q , one actually has: $\rho(CSpin(Q)) = SO(Q)$, ([C], Theorem 10.3.1).

The group $CSpin(Q)$ also acts by multiplication on the left on $C^+(Q)$, this gives a homomorphism, called the spin representation (cf. [S2], 9.3):

$$\sigma : CSpin(Q) \longrightarrow GL(C^+(Q)), \quad g \longmapsto [x \longmapsto gx].$$

We will identify $CSpin(Q)$ with its image in $GL(C^+(Q))$.

6.3. Proposition.

1. The image of the homomorphism $h_s : \mathbf{C}^* \rightarrow GL(C^+(Q)_{\mathbf{R}})$ is contained in the algebraic group $CSpin(Q)(\mathbf{R})$. Hence $MT(C^+(Q)) \subseteq CSpin(Q)$, and

$$MT(C^+(Q)) = CSpin(Q) \quad \text{if} \quad MT(V) = GO(Q).$$

2. For $t, \phi \in \mathbf{R}$ we have:

$$h(te^{i\phi}) = t^2 \rho(h_s(e^{i\phi})),$$

so we recover (V, Q) from the Hodge structure on $C^+(Q)$ defined by h_s .

3. The Hodge structure (V, h) is a sub-Hodge structure of $(C^+(Q) \otimes C^+(Q), h_s \otimes h_s)$:

$$V \hookrightarrow C^+(Q) \otimes C^+(Q).$$

Proof. First we show $h_s(\mathbf{C}^*) \in CSpin(Q)(\mathbf{R})$. Since $h_s(te^{i\phi}) = t^2(a - bJ)$ with $a^2 + b^2 = 1$ and then $(a - bJ)^{-1} = a + bJ$ it suffices to show $(a - bJ)V(a + bJ) \subset V$. For $v \in V_{\mathbf{R}} \subset C(Q)_{\mathbf{R}}$ and $a, b \in \mathbf{R}$ we have:

$$(a - bJ)v(a + bJ) = (a^2v - b^2JvJ) + ab(-Jv + vJ).$$

Recall that $V_{\mathbf{R}} = V_1 \oplus V_2$ (orthogonal sum) and $J = f_1f_2$ with $f_i \in V_2$. Thus $Jv = vJ$ for $v \in V_1$, and hence $(a - bJ)v(a + bJ) = (a^2 + b^2)v \in V$. Note that

$$Jf_1 = (f_1f_2)f_1 = -f_1^2f_2 = f_2 \in V, \quad Jf_2 = (f_1f_2)f_2 = -f_1 \in V.$$

Similarly, $f_1J = -f_2$, $f_2J = f_1$. This gives:

$$(a - bJ)f_1(a + bJ) = (a^2 - b^2)f_1 - 2abf_2, \quad (a - bJ)f_2(a + bJ) = (a^2 - b^2)f_2 + 2abf_1$$

hence also $(a - bJ)V_2(a + bJ) \subset V$ and we conclude that $h_s(\mathbf{C}^*) \in CSpin(Q)(\mathbf{R})$.

Moreover, we see that $\rho(h_s(a + bi))v = v$ for $v \in V_1$ and that $f_1 \pm if_2$ is an eigenvector of $\rho(h_s(a + bi))$ ($\in GL(V_{\mathbf{R}})$) with eigenvalue $(a \pm ib)^2$, this verifies 6.3.2.

We already saw that $h_s(\mathbf{C}^*) \subset CSpin(Q)(\mathbf{R})$, hence $MT(C^+(Q)) \subseteq CSpin(Q)$. If we assume $MT(V) = GO(Q)$ and G is a subgroup of $SO(Q)$ defined over \mathbf{Q} with $h(e^{i\phi}) \in G(\mathbf{R})$

for all $\phi \in \mathbf{R}$, then G must be equal to $SO(Q)$. Hence $\rho(MT(C^+(Q))) = SO(Q)$. Since $\ker(\rho) \cong \mathbf{G}_m (\subset MT(C^+(Q)))$, we get $MT(C^+(Q)) = CSpin(Q)$.

For $g \in CSpin(Q)$ one has $\nu(g) := \iota(g)g \in \mathbf{G}_m$ ([S2], Lemma 9.3.2), hence:

$$E(gv, gw) = Tr(\alpha \iota(v) \iota(g)gw) = \nu(g)E(v, w).$$

Therefore the isomorphism $V \rightarrow V^*$ defined by E is equivariant for the action of $CSpin(Q)$ (up to the homomorphism ν). We will identify the $CSpin(Q)$ -representations $End(C^+(Q)) = C^+(Q)^* \otimes C^+(Q) \cong C^+(Q) \otimes C^+(Q)$.

We choose an invertible element, say e_1 , in $V (\subset C(Q))$. Then we have an inclusion:

$$V \hookrightarrow End(C^+(Q)), \quad v \longmapsto M_v := [y \mapsto vye_1].$$

The image of V in $End(C^+(Q))$ is a sub-representation on which $CSpin(Q)$ acts via ρ :

$$(gM_v g^{-1})(y) = (gM_v)(g^{-1}y) = g(vg^{-1}ye_1) = (gv g^{-1})ye_1 = M_{gv g^{-1}}(y) = M_{\rho(g)v}y.$$

As $h_s(\mathbf{C}^*) \subset CSpin(Q)(\mathbf{R})$ it follows that $V \hookrightarrow C^+(Q) \otimes C^+(Q)$ is sub-Hodge structure. \square

6.4. The spin representation is not irreducible in general. In fact, for any $x, y \in C^+(Q)$ we have $\sigma(g)(xy) = gxy = (\sigma(g)x)y$, so the \mathbf{Q} -linear maps $C^+(Q) \rightarrow C^+(Q)$, $x \mapsto xy$ commute with $CSpin(Q)$. Therefore we have an injective map

$$C^+(Q) \hookrightarrow End_{CSpin}(C^+(Q)) \quad (:= \{M \in End(C^+(Q)) : M\sigma(g) = \sigma(g)M\}).$$

(Due to the action on the right, we should write $C^+(Q)^{op}$, but $C^+(Q)^{op} \cong C^+(Q)$, cf. [La], V, Prop. 1.11.) These are all the maps which commute with the $CSpin(Q)$ representation:

6.5. **Lemma.** We have:

$$C^+(Q) \cong End_{CSpin(Q)}(C^+(Q)).$$

Proof. The group $CSpin(Q)$ is an algebraic subgroup of $GL(C^+(Q))$ and thus its Lie algebra, $cspin(Q)$ is a subalgebra of $End(C^+(Q))$ and $cspin(Q) \otimes_{\mathbf{Q}} \mathbf{C}$ is the Lie algebra of the complex Lie group $CSpin(Q)(\mathbf{C})$. Moreover, $End_{CSpin(Q)}(C^+(Q)) \cong End_{cspin(Q)}(C^+(Q))$. The latter is the subspace of $C^+(Q)$ defined by:

$$End_{cspin(Q)}(C^+(Q)) = \{X \in End(C^+(Q)) : XM - MX = 0 \quad \forall M \in cspin(Q)\}.$$

Considering these equations for $X \in End(C^+(Q)_{\mathbf{C}})$ we see that

$$End_{cspin(Q)}(C^+(Q)) \otimes_{\mathbf{Q}} \mathbf{C} \cong End_{cspin(Q) \otimes_{\mathbf{Q}} \mathbf{C}}(C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}).$$

From the representation theory of complex Lie algebras (see 6.6 below for the case n even) we know that

$$End_{cspin(Q) \otimes_{\mathbf{Q}} \mathbf{C}}(C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}) \cong C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C},$$

we use that $C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C}$ is the even Clifford algebra of the quadratic form Q on $V \otimes_{\mathbf{Q}} \mathbf{C}$. Thus $\dim_{\mathbf{Q}} End_{cspin(Q)}(C^+(Q)) = \dim_{\mathbf{C}} C^+(Q) \otimes_{\mathbf{Q}} \mathbf{C} = 2^{n-1}$, hence $C^+(Q)$ must be all of $End_{cspin(Q)}(C^+(Q))$ for dimension reasons. \square

6.6. Example. In case Q is the quadratic form $Y_1Y_{m+1} + \dots + Y_mY_{2m}$ (this is the case over a finite extension of \mathbf{Q}), the Clifford algebra and the spin representation can be described as follows (cf. [FH], p. 304; they consider the Clifford algebra over \mathbf{C} , but the same arguments work over any extension of \mathbf{Q}).

Let Z be the m -dimensional subspace of V defined by $Y_{m+1} = \dots = Y_{2m} = 0$. Then

$$C(Q) \cong \text{End}(\Lambda^*Z), \quad \Lambda^*Z = \mathbf{Q} \oplus Z \oplus \wedge^2 Z \oplus \dots$$

The even Clifford algebra is identified with the subalgebra

$$C^+(Q) \cong \text{End}(\Lambda^{\text{even}}Z) \times \text{End}(\Lambda^{\text{odd}}Z) \cong M_{2^{m-1}}(\mathbf{C}) \times M_{2^{m-1}}(\mathbf{C}).$$

Since $C^+(Q) \subset \text{End}_{\text{spin}(Q)}(C^+(Q))$, the spin representation must, at least, split in the direct sum of 2^{m-1} copies of a representation σ_+ and 2^{m-1} copies of a representation σ_- . By considering the Lie algebra action one can see that σ_+ and σ_- are both irreducible and are not isomorphic (see [FH]). This implies that $C^+(Q) = \text{End}_{\text{spin}(Q)}(C^+(Q))$. The representations σ_+ and σ_- , each of dimension 2^{m-1} , are the half spin representations of $so(Q)$.

7. CLIFFORD ALGEBRAS

7.1. We recall that the Clifford algebra can be built up from quaternion algebras. If $Q = d_1X_1^2 + d_2X_2^2 + \dots + d_nX_n^2$, with $d_i \in \mathbf{Q}$, we write:

$$V = \langle d_1 \rangle \oplus \dots \oplus \langle d_n \rangle.$$

7.2. Quaternion Algebras. The quaternion algebra $A = (a, b)_F$ over a field F has an F -basis $1, i, j, k$ such that

$$A = F \oplus Fi \oplus Fj \oplus Fk, \quad i^2 = a, \quad j^2 = b, \quad ij = k = -ji, \quad a, b \in F^*,$$

with $F^* = F - \{0\}$ and $xa = ax$ for all $x \in F, a \in A$. In case $F = \mathbf{Q}$ we omit the index F . Note however that $(a, b)_F$ and $(c, d)_F$ may be isomorphic without $a = c$ and $b = d$. For example, $(a, b)_F \cong (b, a)_F$ (via $i \mapsto j, j \mapsto i, k \mapsto -k$). The algebra $M_2(F)$ of 2×2 matrices over F is a quaternion algebra and for any $b \in F^*$

$$M_2(F) \cong (1, b)_F, \quad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

The quaternion algebra $(a, b)_F$ is either a (skew) field or is isomorphic to the matrix algebra $M_2(F)$. One has: $(a, b)_F \cong M_2(F)$ if and only if the equation $ax^2 + by^2 = abz^2$ has a non-trivial solution $(x, y, z) \in F^3$ ([La], III, 2.7; [S2], Corollary 2.11.10; note that if $ax^2 + by^2 - abz^2 = 0$ then $(xi + yj + zk)^2 = 0$ hence $(a, b)_F$ is not a field).

7.3. Example. In case $Q = dX^2, d \in \mathbf{Q}^*$, the Clifford algebra is isomorphic to $\mathbf{Q}[T]/(T^2 - d)$, which is a field if d is not a square and is isomorphic to $\mathbf{Q} \times \mathbf{Q}$ if d is a square. In this case $C^+(Q) \cong \mathbf{Q} = M_1(\mathbf{Q})$.

7.4. Example. In case $Q = aX_1^2 + bX_2^2$, $C(Q)$ is the quaternion algebra (a, b) . The even Clifford algebra $C^+(Q) = \mathbf{Q} \oplus \mathbf{Q}e_1e_2$ with $(e_1e_2)(e_1e_2) = -e_1^2e_2^2 = -ab$, so if $-ab$ is not a square, the even Clifford algebra is a field: $C^+(Q) \cong \mathbf{Q}(\sqrt{-ab})$. In case $-ab = c^2$ is a square, $C^+(Q) \cong \mathbf{Q} \times \mathbf{Q}, e_1e_2 \mapsto (c, -c)$.

7.5. Example. Let $Q = d_1X_1^2 + d_2X_2^2 + d_3X_3^2$, note that $1, i := e_1e_2, j := e_2e_3, k := d_2e_1e_3$, satisfy the rules of the quaternion algebra $(-d_1d_2, -d_2d_3)$ and that $1, \dots, k$ are a basis of $C^+(Q)$ hence

$$C^+(Q) \cong (-d_1d_2, -d_2d_3).$$

The element $z = e_1e_2e_3$ is in the center of the Clifford algebra $C(Q)$ (since it commutes with the generators e_1, e_2, e_3) and $z^2 = -e_1^2e_2^2e_3^2 = -d_1d_2d_3$. Moreover, $C(Q) = C^+(Q) \oplus zC^+(Q)$, hence $C(Q) \cong C^+(Q) \otimes \mathbf{Q}(\sqrt{-d_1d_2d_3})$. Since $\mathbf{Q}(\sqrt{-d_1d_2d_3}) \cong C(-d_1d_2d_3X^2)$, we have ([S2], Lemma 9.2.9, with some changes in notation):

$$C(d_1X_1^2 + d_2X_2^2 + d_3X_3^2) \cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3 \rangle).$$

7.6. Graded tensor products. To determine the Clifford algebra of the quadratic form $Q = d_1X_1^2 + \dots + d_nX_n^2$ on V , so $V = \langle d_1 \rangle \oplus \dots \oplus \langle d_n \rangle$, we use that if $V = V' \oplus V''$ is the orthogonal direct sum with $Q|_{V'} = Q'$ and $Q|_{V''} = Q''$ then ([S2], 9.2.5):

$$C(Q' \oplus Q'') = C(Q') \hat{\otimes} C(Q'').$$

Here $\hat{\otimes}$, the graded tensor product, is the tensor product of the underlying \mathbf{Q} -vector spaces but the product is given by ([S2], 9.1.4)) $(x \hat{\otimes} y)(x' \hat{\otimes} y') = \epsilon xx' \hat{\otimes} yy'$ with $x, x' \in C^\pm(Q')$ and $y, y' \in C^\pm(Q'')$ and $\epsilon = 1$ except if $y \in C^-(Q'')$ and $x' \in C^-(Q')$, in that case $\epsilon = -1$.

An easy example is the case $Q = aX^2 + bY^2$, $Q' = \langle a \rangle$ and $Q'' = \langle b \rangle$, in that case $C(Q' \oplus Q'') = C(Q') \hat{\otimes} C(Q'') \cong (a, b)$ as in example 7.4. Using example 7.5 as induction step we then obtain, from Theorem 9.2.10 of [S2] and a basic fact on quaternion algebras, the following result:

7.7. Theorem. Let $Q = \sum d_iX_i^2$, with $d_i \in \mathbf{Q}^*$. Then we have:

1. In case $n = 2m$, let $d := (-1)^m d_1 \dots d_n$. Then the even Clifford algebra $C^+(Q)$ is isomorphic to one of the following two algebras
 - (a) In case $\sqrt{d} \in \mathbf{Q}$ we have $C^+(Q) \cong M_{2m-2}(D) \times M_{2m-2}(D)$ with a quaternion algebra D over \mathbf{Q} .
 - (b) In case $\sqrt{d} \notin \mathbf{Q}$ we have $C^+(Q) \cong M_{2m-2}(D) = M_{2m-2}(\mathbf{Q}) \otimes_{\mathbf{Q}} D$, where D is a quaternion algebra over a quadratic field extension $F = \mathbf{Q}(\sqrt{d})$ of \mathbf{Q} .
2. In case $n = 2m + 1$, we have $C^+(Q) \cong M_{2m-1}(D)$ for a quaternion algebra D over \mathbf{Q} .

Note that if $D \cong M_2(\mathbf{Q})$ then $M_{2^a}(D) \cong M_{2^{a+1}}(\mathbf{Q})$.

Proof. For $n \leq 3$ see 7.3, 7.4, 7.5. For $n > 3$ use Example 7.5:

$$\begin{aligned} C(\langle d_1 \rangle \oplus \dots \oplus \langle d_n \rangle) &\cong C(\langle d_1 \rangle \oplus \dots \oplus \langle d_3 \rangle) \hat{\otimes} C(\langle d_4 \rangle \oplus \dots \oplus \langle d_n \rangle) \\ &\cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3 \rangle) \hat{\otimes} C(\langle d_4 \rangle \oplus \dots \oplus \langle d_n \rangle) \\ &\cong (-d_1d_2, -d_2d_3) \otimes C(\langle -d_1d_2d_3 \rangle \oplus \langle d_4 \rangle \oplus \dots \oplus \langle d_n \rangle). \end{aligned}$$

in the last lines the first tensor product is the usual one since all elements in $(-d_1d_2, -d_2d_3)$ are even elements of $C(\langle d_1 \rangle \oplus \langle d_2 \rangle \oplus \langle d_3 \rangle)$.

In case $n = 4$ one thus finds $C(Q) \cong (-d_1d_2, -d_2d_3) \otimes (-d_1d_2d_3, d_4)$ and $C^+(Q) \cong (-d_1d_2, -d_2d_3) \otimes F$, with $F \cong \mathbf{Q}[X]/(X^2 - d_1d_2d_3d_4)$ the even Clifford algebra of $\langle -d_1d_2d_3 \rangle \oplus \langle d_4 \rangle$. So F is either a field or is isomorphic to $\mathbf{Q} \times \mathbf{Q}$.

We continue this game and we find quaternion algebras $A_{13}, A_{15}, \dots, A_{1,2k+1}$, such that

$$C(Q) \cong A_{13} \otimes A_{15} \otimes \dots \otimes A_{1,2k+1} \otimes C(\langle (-1)^k d_1d_2 \dots d_{2k+1} \rangle \oplus \langle d_{2k+2} \rangle \oplus \dots \oplus \langle d_n \rangle),$$

and the even Clifford algebra is:

$$C^+(Q) \cong A_{13} \otimes A_{15} \otimes \dots A_{1,2k+1} \otimes C^+(\langle (-1)^k d_1 d_2 \dots d_{2k+1} \rangle \oplus \langle d_{2k+2} \rangle \oplus \dots \langle d_n \rangle).$$

We conclude that if $n = 2m$, then $C(Q)$ is the tensor product of $m - 1$ quaternion algebras which is then graded tensored with an m -th quaternion algebra. Thus $C^+(Q)$ is a tensor product of $m - 1$ quaternion algebras with an algebra $\mathbf{Q}[X]/(X^2 - d)$ and it is not hard to see that $d = (-1)^m d_1 \dots d_{2m}$.

In case $n = 2m + 1$, $C(Q)$ is the tensor product of $2m$ quaternion algebras (which is $C^+(Q)$) graded tensored by $C(\langle (-1)^m d_1 \dots d_{2m+1} \rangle)$.

Now we recall that the tensor product $A \otimes_{\mathbf{Q}} B$ of two quaternion algebras A, B over \mathbf{Q} is isomorphic with $M_2(D) = M_2(\mathbf{Q}) \otimes_{\mathbf{Q}} D$ for some quaternion algebra D . Let $A = (a, c)$, $B = (b, d)$ then, in A , we have

$$(x_1 i + x_2 j + x_3 k)^2 = ax_1^2 + cx_2^2 + acx_3^2.$$

The quadratic form in 6 variables $ax_1^2 + cx_2^2 + acx_3^2 - (by_1^2 + dy_2^2 + bdy_3^2)$ is indefinite for any choice of signs for a, \dots, d . Hence Meyer's theorem ([S5], Corollary 4.3.2) implies that it has a non-trivial zero. Thus there are $x \in A, y \in B$ with $x^2 = y^2 = e$, so if A and B are fields they have the quadratic field $K = \mathbf{Q}(\sqrt{e})$ in common. Then A and B are cyclic algebra's over K : $A = (K/\mathbf{Q}, r)$, $B = (K/\mathbf{Q}, s)$ (so $A = K \oplus K\alpha$ with $\alpha^2 = r$ and $\alpha x = \bar{x}\alpha$ where $\bar{}$ is the conjugation on K). In [La], III, 2.11 and [S2], Theorem 8.12.7 one finds an explicit isomorphism $A \otimes_{\mathbf{Q}} B \cong M_2(D)$ with $D = (K/\mathbf{Q}, rs)$. \square

8. KUGA-SATAKE VARIETIES

8.1. The rational, polarized, weight one Hodge structure $(C^+(Q), h_s, E)$ defines an isogeny class of abelian varieties. Each variety in this isogeny class is called a Kuga-Satake variety for (V, h, Q) . More precisely, consider the dual W of the 2^{n-2} -dimensional complex vector space $C^+(Q)^{1,0}$:

$$W := (C^+(Q)^{1,0})^*, \quad \text{let } \Gamma \subset C^+(Q)^*$$

be a free \mathbf{Z} -module with $\Gamma \otimes_{\mathbf{Z}} \mathbf{Q} = C^+(Q)^*$, the dual of the \mathbf{Q} -vector space $C^+(Q)$. The image of Γ under the projection from $(C^+(Q)^{0,1})^*$: $\Gamma \subset C^+(Q)^*_{\mathbf{C}} \rightarrow W$ is a lattice in the complex vector space W and the quotient is an abelian variety A_{Γ} (with Riemann form E_{Γ} defined by E), which is a Kuga-Satake variety of (V, h, Q) . There is a natural isomorphism of rational, polarized, weight one Hodge structures:

$$(H^1(A_{\Gamma}, \mathbf{Q}), E_{\Gamma}) \cong (C^+(Q), h_s, E).$$

For an abelian variety A one has $\text{End}(A) \otimes \mathbf{Q} \cong \text{End}_{\text{Hod}}(H^1(A, \mathbf{Q}))$ and if

$$\text{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q} \cong M_{n_1}(D_1) \times \dots \times M_{n_d}(D_d) \quad \text{then} \quad A \sim A_1^{n_1} \times \dots \times A_d^{n_d}$$

where the D_i are (skew) fields, \sim means isogeneous and the A_i are simple abelian varieties. Thus Theorem 7.7 gives the decomposition in simple factors of a Kuga-Satake variety in case $MT(V) = GO(Q)$.

8.2. Example. Let (V, h, Q) be a three dimensional rational Hodge structure of weight two with $\dim V^{2,0} = 1$ and $MT(V) = GO(Q)$. Assume that with respect to some basis of V we can write $Q = -X_1^2 - X_2^2 + dX_3^2$. Then from Example 7.5 we know:

$$C^+(Q) \cong (-1, d).$$

A Kuga-Satake variety of (V, h, Q) is an abelian surface X with $End(X) \cong C^+(Q)$. Hence it is a product of two isogeneous elliptic curves if $C^+(Q)$ is not a field, otherwise it is a simple abelian surface. Note that $C^+(Q) \cong M_2(\mathbf{Q})$ iff $-x^2 + dy^2 = -dz^2$ has a non-trivial solution in \mathbf{Q}^3 . This happens for example if $d = 1$. In case $d \equiv 3 \pmod{4}$ we get a field, this is best seen by multiplying the equation $-x^2 + dy^2 = -dz^2$ by d and substituting $y := d^{-1}y$, $z := d^{-1}z$ to get $dx^2 = y^2 + z^2$. If this had a non-trivial solution, we had one in \mathbf{Z}^3 (multiply by product of denominators) and after dividing by a power of 2 at least one of x, y, z would be odd. Now consider the equation mod 4 and use that a square is either 0 or 1 mod 4.

8.3. Geometric realizations. In 4.5 we observed that given (V, Q) , one can find weight two Hodge structures (V, h, Q) . Since $V \subset C^+(Q) \otimes C^+(Q)$ we find an inclusion of Hodge structures:

$$V \hookrightarrow H^1(A_\Gamma, \mathbf{Q}) \otimes H^1(A_\Gamma, \mathbf{Q}) \subset H^2(A_\Gamma \times A_\Gamma, \mathbf{Q}).$$

Thus any polarized weight two Hodge structure (V, h, Q) with $\dim V^{2,0} = 1$ is a sub-Hodge structure of the cohomology of some algebraic variety.

This is *not* true if $\dim V^{2,0} > 1$, Griffiths work on variations of Hodge structures (‘Griffiths transversality’) implies that the general polarized Hodge structure of weight two (and similar results hold for higher weight) with $\dim H^{2,0} > 1$ is not a sub-Hodge structure of the cohomology of an algebraic variety, see [CKT] and [Ma].

9. ABELIAN VARIETIES OF WEIL TYPE AND KUGA-SATAKE VARIETIES

9.1. The Hodge conjecture for abelian varieties of dimension at least 4 is still open (see [G] for a recent overview). For 4 dimensional abelian varieties (abelian 4-folds) Moonen and Zarhin [MZ1], [MZ2] proved that if the Hodge conjecture is true for abelian 4-folds of Weil type, then the Hodge conjecture is true for all abelian 4-folds.

An abelian $2n$ -fold A is of Weil type if $End(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ contains an imaginary quadratic field $K = \mathbf{Q}(\sqrt{-d})$, $d > 0$ and the action of the endomorphism $\sqrt{-d}$ on the tangent space at the origin of A has eigenvalues $\sqrt{-d}$, $-\sqrt{-d}$, each with multiplicity n . Associated to the pair (A, K) is a ‘discriminant’ $\delta \in \mathbf{Q}^*/N(K^*)$ where $N(K^*)$ is the group of the norms of K^* , that is, the elements $a^2 + b^2d$, $a, b \in \mathbf{Q}$ (see [vG] for an introduction to these varieties).

The Hodge conjecture for the general abelian $2n$ -fold of Weil type with $d = 3$, $\delta = 1$ is proved by C. Schoen in [S3] for $n = 2$. The case $n = 3$ is done in [S5]. There he also proves that the Hodge conjecture for the general abelian 4-fold of Weil type with $d = 3$ (and any δ) follows from this by specializing the 6-fold to a product of a 4-fold and an abelian surface.

The following result, due to G. Lombardo, may also be of some interest. The case that $K = \mathbf{Q}(\sqrt{-1})$ was considered by Paranjape [P] and, from a different point of view, by Matsumoto and others ([M] and references given there).

9.2. Theorem. Let $(A, K = \mathbf{Q}(\sqrt{-d}))$ be an abelian 4-fold of Weil type with discriminant $\delta = 1$. Then A^4 is a Kuga-Satake variety of a weight two Hodge structure (V, h, Q) with

$$Q = -X_1^2 - X_2^2 + X_3^2 + X_4^2 + X_5^2 + dX_6^2.$$

Conversely, if (V, h, Q) is a weight two polarized Hodge structure with Q as above, then the Kuga-Satake variety of (V, h, Q) is isogeneous to A^4 with A an abelian 4-fold of Weil type.

Proof. See [Lo], we only verify that the Kuga-Satake variety of a general (V, h, Q) is isogeneous to A^4 with A an abelian 4-fold with $K = \text{End}(A)$. Using the rules from the proof of Theorem 7.7, one finds:

$$C(Q) \cong (-1, 1) \otimes C(-X^2 + X_4^2 + X_5^2 + dX_6^2) \cong M_2(\mathbf{Q}) \otimes C(-X^2 + X_4^2 + X_5^2 + dX_6^2),$$

since $(-1, 1) \cong (1, -1) \cong M_2(\mathbf{Q})$, cf. 7.2. Next:

$$C(-X^2 + X_4^2 + X_5^2 + dX_6^2) \cong (1, 1) \otimes (C(\langle 1 \rangle) \hat{\otimes} C(\langle d \rangle)) \cong M_2(\mathbf{Q}) \otimes C(X^2 + dY^2),$$

since $C^+(X^2 + dY^2) \cong \mathbf{Q}(\sqrt{-d}) \cong K$ we get:

$$C^+(Q) \cong M_2(\mathbf{Q}) \otimes M_2(\mathbf{Q}) \otimes C^+(X^2 + dY^2) \cong M_4(\mathbf{Q}) \otimes K.$$

Hence for general (V, h, Q) any Kuga-Satake variety is isogeneous to 4 copies of a simple abelian variety A , so $\dim A = 4$, with $\text{End}(A) \cong K$. \square

10. THE KUGA-SATAKE-HODGE CONJECTURE

10.1. Given a polarized Hodge structure (V, h, Q) of weight two with $\dim V^{2,0} = 1$ there exists an abelian variety A , the Kuga-Satake variety of (V, h, Q) , with the property that V is a sub-Hodge structure of $H^2(A \times A, \mathbf{Q})$, cf. section 8.3. In case there is another algebraic variety X with $V \hookrightarrow H^2(X, \mathbf{Q})$, the Hodge conjecture predicts (cf. section 2.6) the existence of an algebraic cycle $Z \subset A^2 \times X$, the Kuga-Satake-Deligne correspondence, which realizes the morphism of Hodge structures

$$f : H^2(A \times A, \mathbf{Q}) \xrightarrow{\pi} V \xrightarrow{i} H^2(X, \mathbf{Q}).$$

(More generally, one can consider $V \hookrightarrow H^{2+2n}(X, \mathbf{Q})(n)$.)

A particular case where this happens is when the algebraic variety X has $\dim H^{2,0}(X, \mathbf{Q}) = 1$. The space of the Hodge cycles $B(H^2(X, \mathbf{Q}))$ a sub-Hodge structure of $H^2(X, \mathbf{Q})$, it is the image of $CH^1(X)_{\mathbf{Q}}$ and is called the Neron-Severi group of X (tensoring by \mathbf{Q}). Let V be the orthogonal complement in $H^2(X, \mathbf{Q})$:

$$H^2(X, \mathbf{Q}) = V \oplus NS(X)_{\mathbf{Q}}.$$

Then V , with the polarization Q induced by the one on $H^2(X, \mathbf{Q})$, is of the type we consider. It has the additional convenient property that it is simple in the sense that if $W \subset V$ is a sub-Hodge structure, then $W = 0$ or $W = V$. In fact, $V = W \oplus W^\perp$ and if $W^{2,0} = 0$ then $W \subset NS(X)_{\mathbf{Q}} \cap V = 0$, else $(W^\perp)^{2,0} = 0$ so $W^\perp = 0$ and $W = V$.

Since for an ample divisor Y on a variety X the restriction map $H^2(X, \mathbf{Q}) \hookrightarrow H^2(Y, \mathbf{Q})$ is injective if $\dim Y \geq 2$ (Lefschetz Theorem), we may assume without loss of generality that X is a surface. Then the Hodge conjecture leads to:

10.2. Kuga-Satake-Hodge conjecture. Let X be a smooth surface with $\dim H^{2,0}(X) = 1$ and let $H^2(X, \mathbf{Q}) = V \oplus NS(X)_{\mathbf{Q}}$. Let Q be the induced polarization on V and let A be a Kuga-Satake variety of (V, h, Q) .

Then there exist a surface Z and a diagram

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & A \times A \\ \pi \downarrow & & \\ X & & \end{array} \quad \text{such that} \quad \pi_* \phi^* : H^2(A \times A, \mathbf{Q}) \longrightarrow H^2(X, \mathbf{Q})$$

induces an isomorphism $V \xrightarrow{\cong} V$.

10.3. We show that conjecture 10.2 is indeed a special case of the Hodge conjecture. Let $f : H^2(A^2, \mathbf{Q}) \rightarrow H^2(X, \mathbf{Q})$ be the morphism of Hodge structures as in 10.1. Then $f \in B(H^{2d}(A^2 \times X, \mathbf{Q}))$ with $d = \dim A^2$ (cf. 2.6). The Hodge conjecture implies that f should be the class of a codimension d cycle $\sum_i a_i Z_i$ on the $d + 2$ -dimensional variety $A^2 \times X$, hence each Z_i is a surface (an irreducible 2 dimensional variety). Each Z_i defines a morphism of Hodge structures $[Z_i] : H^2(A^2, \mathbf{Q}) \rightarrow H^2(X, \mathbf{Q})$, since V is simple the restriction of each $[Z_i]$ to V is either 0 or is an isomorphism on its image. As f is an isomorphism, there is at least one Z_i such that $[Z_i]$ induces an isomorphism, take Z to be that Z_i .

10.4. Assume that Z exists. It is easy to see that we may replace Z by its desingularization. The map $\pi : Z \rightarrow X$ must be surjective (because $H^{2,0}(X) \subset \pi_* H^2(Z, \mathbf{C})$). Since $H^2(A^2, \mathbf{Q}) = \Lambda^2 H^1(A^2, \mathbf{Q})$ and pull-back is compatible with cup product, we get:

$$V \hookrightarrow \pi_* \text{Image} \left(\Lambda^2 H^1(Z, \mathbf{Q}) \longrightarrow H^2(Z, \mathbf{Q}) \right).$$

Conversely, given a surface Z with a surjective map $\pi : Z \rightarrow X$ having the above property, the albanese map $Z \rightarrow \text{Alb}(Z)$ is essentially a map $\phi : Z \rightarrow A^2$ with $\pi_* \phi^*$ as in 10.2.

In the example of Paranjape [P] and in Example 11.3 below the surface Z is a product of two curves, also in the example of Voisin [V] it seems to be possible to choose for Z a product of curves. In all these examples the surfaces X are K3 surfaces (so $\dim H^2(X, \mathbf{Q}) = 22$, $\dim H^{2,0}(X) = 1$), but their Neron-Severi groups are rather large.

11. HODGE STRUCTURES AND IMAGINARY QUADRATIC FIELDS.

11.1. The example of C. Voisin [V] deals with a polarized weight two Hodge structure (V, h, Q) with $\dim V^{2,0} = 1$ which has an automorphism ϕ of order 3 preserving the polarization: $Q(v, w) = Q(\phi v, \phi w)$ (it is induced from an automorphism of a K3 surface). The action of ϕ on V gives V the structure of a vector space over $K = \mathbf{Q}(\sqrt{-3})$.

Voisin asked for the simple factors of the Kuga-Satake variety of (V, h, Q) . She already proved that the (isogeny class of an) elliptic curve A_0 with complex multiplication by K and an abelian variety A_1 with $2 \dim A_1 = \dim V$, $K \subset \text{End}(A_1) \otimes \mathbf{Q}$, are simple components. The following theorem (an exercise in representation theory) gives the simple factors in general.

11.2. Theorem. Let (V, h, Q) be a polarized weight two Hodge structure with $\dim V^{2,0} = 1$. Let $K \subset \text{End}_{\text{Hod}}(V)$ be an imaginary quadratic field such that V is K -vector space and assume $Q(xv, xw) = x\bar{x}Q(v, w)$ for $x \in K$, $v, w \in V$. Let $n = 2m = \dim V$.

Then there is a Hodge structure S such that

$$C^+(Q) \cong S^{2^{m-2}}, \quad \dim_{\mathbf{Q}} S = 2^{m+1}.$$

The Hodge structure S splits as:

$$S \cong S_0 \oplus S_1 \oplus \dots \oplus S_m, \quad S_i \cong S_{m-i}, \quad \dim_{\mathbf{Q}} S_i = 2 \binom{m}{i}.$$

The S_i are simple Hodge structures except S_l if $2l = m$, $l \equiv 2 \pmod{4}$ and the polarization satisfies an additional condition, in that case $S_l \cong (S'_l)^2$ and S'_l is irreducible.

The S_i are K -vector spaces, $K \subset \text{End}_{\text{Hod}}(S_i)$, they are simple if h is general and $V \hookrightarrow S_0 \otimes S_1 \subset C^+(Q) \otimes C^+(Q)$.

Proof. To appear. □

11.3. Example. (With thanks to M. Nori.) Smooth quartic surfaces in \mathbf{P}^3 are K3 surfaces. Consider a surface X in \mathbf{P}^3 defined by:

$$X : \quad T^4 = F(X, Y, Z)$$

where $F = 0$ defines a smooth plane quartic curve $C \subset \mathbf{P}^2$. Let $p \in C$ and let $L_p \subset \mathbf{P}^2$ be the tangent line to C at p . Then $L_p \cap C = \{p, p', p''\}$ (since p has multiplicity 2). If $p \in C$ is general and u is a suitable parameter along L_p , then restriction of F to L_p is given by $u^2(u-1)(u+1)$ (with $u = 0$ corresponding to $p \in L_p$). The curve in X lying over L_p is defined by $T^4 = u^2(u^2-1)$, thus it is irreducible and its normalization E_p has a $4 : 1$ map onto $L_p \cong \mathbf{P}^1$ which is totally ramified over $u = \pm 1$ and over $u = 0$ it has two ramification points. Hence E_p is an elliptic curve and since it has an isomorphism of order 4, $T \mapsto iT$, $i^2 = -1$, with fixed points (the points over $u = \pm 1$) one has $E_p \cong E_i := \mathbf{C}/(\mathbf{Z} + i\mathbf{Z})$. The elliptic surface

$$\mathcal{E} \longrightarrow C, \quad \mathcal{E} := \cup_{p \in C} E_p \times \{p\} \subset \mathbf{P}^3 \times C,$$

is thus isotrivial and projection on the first factor gives a surjective map $\mathcal{E} \rightarrow X$. After a suitable base change $C' \rightarrow C$ and normalization the surface will be a product $E_i \times C'$. Thus we have a surjective rational map:

$$\pi : E_i \times C' \longrightarrow X,$$

so $V \subset H^2(X, \mathbf{Q})$ lies in the image of π_* . Now $S_0 \cong H^1(E_i, \mathbf{Q})$, hence E_i is isogeneous to a simple factor A_0 of the Kuga-Satake variety A of $V \subset H^2(X, \mathbf{Q})$ and one verifies that $S_1 \subset H^1(C', \mathbf{Q})$, hence $\text{Jac}(C')$ maps onto a simple factor A_1 of A . Then we may take $Z = E_i \times C'$ (or a blow up if π is not a morphism), the map ϕ is the composition:

$$\phi : E_i \times C' \longrightarrow E' \times \text{Jac}(C') \longrightarrow A_0 \times A_1 \hookrightarrow A \times A.$$

11.4. The following result was inspired by [V] (and was generalized following a discussion with M. Nori). It has interesting geometrical applications and gives a better understanding of the Hodge structures S_i .

11.5. Theorem. Let (V, h, Ψ) be a polarized Hodge structure of weight k , let $K \subset \text{End}_{\text{Hod}}(V)$ be an imaginary quadratic field such that V is a K -vector space and assume $\Psi(xv, xw) = x\bar{x}\Psi(v, w)$ for $x \in K$, $v, w \in V$.

If K acts via scalar multiplication on $V^{k,0}$, then there exist (V, h', Ψ') , (K, h_K, E_K) , polarized Hodge structures of weight $k - 1$ and one respectively, such that

$$(V, h) \hookrightarrow (V, h') \otimes (K, h_K), \quad \text{and} \quad \Psi = (\Psi' \otimes E_K)|_V.$$

Proof. A generalization, with K a CM-field, is to appear. □

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